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Research Article

Static Electric Force and Measurement Principle of Material Constants in Electrostrictive Material

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Electrostrictive materials convert electrical energy into mechanical energy and vice versa. They are extensive applied as intelligent materials in the engineering structures. The governing equations in electrostrictive media under the quasistatic electric field are very important for the measurement of material constants and the research on the strength and function. But some theoretical problems should be further clarified. In this paper, the electric force acting on the material is studied and the complete governing equations will be given. In this paper a possible method to measure electrostrictive coefficients is also discussed.

1. Introduction

The measurement method of material constants in an electrostrictive material is somewhat controversial between authors. Shkel and Klingenberg [1] considered that "The ultimate deformation depends on the elastic properties of the fixtures attached to the material (e.g., the electrodes). The (electrostrictive) coefficients γ_{ijkl} are therefore not strictly material parameters, but rather characteristics of the entire system." Zhang et al. [2] pointed out that "In general in a nonpiezoelectric material such as the polyurethane elastomers investigated, the electric field induced strain can be caused by the electrostrictive effect and also by the Maxwell stress effect. The electrostrictive effect is the direct coupling between the polarization and mechanical response in the material. … On the other hand Maxwell stress, which is due to the interaction between the free charges on the electrodes (Coulomb interaction) and to electrostatic forces that arise from dielectric inhomogeneities." Guillot et al. [3] considered that "strictly speaking, the Maxwell stress tensor does not belong to the electrostrictive equations, but that it should be taken into account in the measurements. … it is possible to factor out its contribution to the total response of the film and therefore to identify the isolated contribution due to the (electrostrictive) tensor only." Thakur and Singh [4] considered that: "In most of the recent experiments concerning determination of electrostrictive parameters in elastic dielectrics, several researchers used incorrect equations without considering the contribution from the edge effect, the shear stress and suitable boundary conditions. This led to wrong predictions of experimental results particularly for materials with high Poisson ratios. Errors in the estimation of induced strains, varying from an underestimation of 202% to an overestimation of 168%, have been pointed out in the case of polycarbonate (PC)." Some material scientists felt puzzled about the objectivity of the electrostrictive coefficients.

In the books of Stratton [5] and Landau and Lifshitz [6], the formula of the stress in an isotropic electrostrictive material is

$$\sigma_{ik} = \frac{\partial g_0}{\partial \varepsilon_{ik}} + \sigma_{ik}^L,$$
$$\sigma_{ik}^L = \left(\frac{1}{2}\right)(2\epsilon - a_1)E_k E_i - \left(\frac{1}{2}\right)(\epsilon + a_2)E_m E_m \delta_{ik}, \tag{1}$$

where $\partial g_0/\partial \varepsilon_{ik}$ is the stress without the electromagnetic field in the medium. As shown in [7, 8] σ_{ik}^L in this formula is just the pseudo total stress [9, 10] which is the sum of the Maxwell stress and the Cauchy stress introduced by the constitutive equations. In Pao's paper [11], he considered that the expression of the Maxwell stress is not unique and he gave some different expressions by different authors. McMeeking and Landis [12] considered that "Since there are no experiments that can separate the effects of the Cauchy and Maxwell stresses unambiguously, it is generally more profitable to consider their sum and not to try identify them separately," "we obviate the need to develop a constitutive theory that is consistent with a pre-determined formulation of the Maxwell stress, as can often be found in the literature on electrostrictive materials. Instead, the constitutive model can be simplified to one that embraces simultaneously the Cauchy, Maxwell, electrostrictive and electrostatic stresses, which in any case cannot be separately identified from any experiment."

From the above and other literatures we can find that different author has different understanding about the governing equations and the Maxwell stress for the electrostrictive materials. We offered the physical variational principle [7, 8, 13–17] based on the thermodynamics in the nonlinear electroelastic analysis to get the governing equations and get the Maxwell stress naturally. The strength problem in engineering is determined by the Cauchy stress, which is connected with the constitutive equation, and the Maxwell stress is an external effective static Coulomb electric force. In this paper we give a general method to determine the static electric force or the Maxwell stress acting on the material by using the migratory variation of φ in the energy principle. We are sorry that we do not have the ground to do experiments, so in this paper only a possible method to measure electrostrictive coefficients is discussed.

2. The Physical Variational Principle

In literatures [7, 8, 13–17] we proposed that the first law of thermodynamics includes two contents: energy conservation law and physical variational principle (PVP), that is,

$$\begin{aligned}&\text{Classical Energy conservation}: \int_V du\, dV - dw - dQ = 0,\\&\text{Classical linear PVP}: \delta\Pi = \int_V \delta u\, dV - \delta w - \delta Q = 0.\end{aligned} \tag{2}$$

We proposed the physical variational principle as a basic principle in the continuum mechanics. From this principle we get the governing equations of the nonlinear electroelastic materials. For this principle with the electric Gibbs function we can simply illustrated as follows.

Under the small deformation the electric Gibbs function or electric enthalpy $g = U - E_i D_i$, where U is the internal energy, can be expanded in the series of $\boldsymbol{\varepsilon}$ and $\mathbf{E}$:

$$\begin{aligned}g = {}&\left(\frac{1}{2}\right) C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \left(\frac{1}{2}\right)\epsilon_{kl}E_kE_l - e_{kij}E_k\varepsilon_{ij}\\&-\left(\frac{1}{2}\right) l_{ijkl}E_iE_j\varepsilon_{kl},\end{aligned}$$

$$\begin{aligned}&C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij},\\&l_{ijkl} = l_{jikl} = l_{ijlk} = l_{klij},\\&e_{kij} = e_{kji}, \qquad \epsilon_{kl} = \epsilon_{lk}.\end{aligned} \tag{3}$$

The constitutive equations are

$$\begin{aligned}\sigma_{kl} &= \frac{\partial g}{\partial \varepsilon_{kl}} = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j - \left(\frac{1}{2}\right) l_{ijkl}E_iE_j,\\D_k &= -\frac{\partial g}{\partial E_k} = \left(\epsilon_{kl} + l_{ijkl}\varepsilon_{ij}\right)E_l + e_{kij}\varepsilon_{ij} \approx \epsilon_{kl}E_l.\end{aligned} \tag{4}$$

In (3) and (4) $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{D}$, $\mathbf{E}$ are the stress, strain, electric displacement, and electric field, respectively, $\mathbf{C}$, $\mathbf{e}$, $\boldsymbol{\epsilon}$, $\mathbf{l}$ are the elastic coefficient, piezoelectric coefficient, permittivity, and the electrostrictive coefficient, respectively. Using (4), (3) is reduced to

$$\begin{aligned}g &= \left(\frac{1}{2}\right) C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} + g^e,\\g^e &= -\left(\frac{1}{2}\right)\left(D_kE_k + \Delta_{kl}\varepsilon_{lk}\right), \quad \Delta_{kl} = e_{mkl}E_{mk},\end{aligned} \tag{5}$$

where g^e is the part of g related to the electric field. The value $\boldsymbol{\Delta} : \boldsymbol{\varepsilon}$ in (5) can be neglected due to its very small. In the electroelastic analysis the dielectric, its environment and their common boundary a^{int} consociate a system and should be considered together, because the electric field exists in every material except the ideal conductor. In this paper the variables in the environment will be denoted by a right superscript "env," and the variables on the interface will be denoted by a right superscript "int." In the environment (3)–(5) are all held.

Under the assumption that $\mathbf{u}$, φ, $\mathbf{u}^{\text{env}}$, φ^{env} satisfy their boundary conditions on their own boundaries a_u, a_φ, a_u^{env}, a_φ^{env}, and the continuity conditions on the interface a^{int}. The physical variational principle with the electric Gibbs free energy is [7, 8, 13–17]:

$$\begin{aligned}&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0,\\&\delta\Pi_1 = \int_V \delta g\, dV + \int_V g^e \delta u_{i,i} dV - \delta W,\\&\delta\Pi_2 = \int_{V^{\text{env}}} \delta g^{\text{env}}\, dV + \int_{V^{\text{env}}} g^{e\,\text{env}} \delta u_{i,i}^{\text{env}}\, dV - \delta W^{\text{env}},\\&\delta W = \int_V \left(f_k - \rho\ddot{u}_k\right)\delta u_k\, dV - \int_V \rho_e \delta\varphi\, dV\\&\qquad + \int_{a_\sigma} T_k^* \delta u_k\, da - \int_{a_D} \sigma^* \delta\varphi\, da,\\&\delta W^{\text{env}} = \int_{V^{\text{env}}} \left(f_k^{\text{env}} - \rho\ddot{u}_k^{\text{env}}\right)\delta u_k^{\text{env}}\, dV - \int_{V^{\text{env}}} \rho_e^{\text{env}} \delta\varphi^{\text{env}}\, dV\\&\qquad + \int_{a_\sigma^{\text{env}}} T_k^{*\text{env}} \delta u_k^{\text{env}}\, da - \int_{a_D^{\text{env}}} \sigma^{*\text{env}} \delta\varphi^{\text{env}}\, da,\\&\delta W^{\text{int}} = \int_{a^{\text{int}}} T_k^{*\,\text{int}} \delta u_k\, da - \int_{a^{\text{int}}} \sigma^{*\,\text{int}} \delta\varphi\, da,\end{aligned} \tag{6}$$

where $\mathbf{f}$, $\mathbf{T}^*$, σ^*, are given body force per volume, surface traction per area, and surface electric charge density, $\mathbf{f}^{\text{env}}$, $\mathbf{T}^{*\text{env}}$, $\sigma^{*\text{env}}$, and $\mathbf{T}^{*\,\text{int}}$, $\sigma^{*\,\text{int}}$ are also given values in the environment and on the interface, respectively. $\mathbf{n} = -\mathbf{n}^{\text{env}}$ is the outward normal on the interface of the body.

The variation of the virtual electric potential φ is divided into local variation $\delta_\varphi \varphi$ and migratory variation $\delta_u \varphi$, and the similar divisions in $\mathbf{E}$, so we have

$$\delta\varphi = \delta_\varphi\varphi + \delta_u\varphi, \qquad \delta_u\varphi = \varphi_{,p}\delta u_p = -E_p\delta u_p,$$
$$\delta E_i = \delta_\varphi E_i + \delta_u E_i, \qquad \delta_\varphi E_i = -\delta_\varphi \varphi_{,i}, \qquad \delta_u E_i = E_{i,p}\delta u_p. \tag{7}$$

The derivation of $\delta\mathbf{E}$ is in Appendix A. Finishing the variational calculation finally we get the governing equations:

$$\begin{aligned}
&S_{jk,j} + f_k = \rho\ddot{u}_k, \quad D_{i,i} = \rho_e, \quad \text{in } V,\\
&S_{jk}n_j = T_k^*, \quad \text{on } a_\sigma, \qquad D_i n_i = -\sigma^*, \quad \text{on } a_D,\\
&\sigma_{ik}^M = D_iE_k - \left(\frac{1}{2}\right)D_nE_n\delta_{ik},\\
&S_{kl} = \sigma_{kl} + \sigma_{kl}^M = C_{ijkl}\varepsilon_{ij} - e_{jkl}E_j\\
&\qquad - \left(\frac{1}{2}\right)l_{ijkl}E_iE_j + D_iE_k - \left(\frac{1}{2}\right)D_nE_n\delta_{ik},
\end{aligned} \tag{8}$$

where σ_{ik}^M is the Maxwell stress and S_{kl} is the pseudo total stress [9, 10]. In the environment we have

$$\begin{aligned}
&S_{ij,i}^{\text{env}} + f_j^{\text{env}} = \rho^{\text{env}}\ddot{u}_j^{\text{env}}, \quad D_{i,i}^{\text{env}} = \rho_e^{\text{env}}, \quad \text{in } V^{\text{env}},\\
&S_{ij}^{\text{env}}n_i^{\text{env}} = T_j^{*\text{env}}, \quad \text{on } a_\sigma^{\text{env}},\\
&D_i^{\text{env}}n_i^{\text{env}} = -\sigma^{*\text{env}}, \quad \text{on } a_D^{\text{env}}.
\end{aligned} \tag{9}$$

On the interface we have

$$\left(S_{ij} - S_{ij}^{\text{env}}\right)n_i = T_j^{*\,\text{int}}, \quad \left(D_i - D_i^{\text{env}}\right)n_i = -\sigma^{*\,\text{int}}, \quad \text{on } a^{\text{int}}. \tag{10}$$

The above variational principle requests prior that the displacements, the electric potential satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations:

$$\begin{aligned}
&u_i = u_i^*, \quad \text{on } a_u, \qquad \varphi = \varphi^*, \quad \text{on } a_\varphi,\\
&u_i^{\text{env}} = u_i^{*\text{env}}, \quad \text{on } a_u^{\text{env}}, \qquad \varphi^{\text{env}} = \varphi^{*\text{env}}, \quad \text{on } a_\varphi^{\text{env}},\\
&u_i = u_i^{\text{env}}, \quad \varphi = \varphi^{\text{env}}, \quad \text{on } a^{\text{int}}.
\end{aligned} \tag{11}$$

Equations (8)–(11) are the governing equations in the electroelastic analysis.

If we introduce the body electric force f_k^e, $f_k^{e\,\text{env}}$, and surface electric force T_k^e, $T_k^{e\,\text{env}}$ in the materials as

$$\begin{aligned}
&f_k^e = \sigma_{jk,j}^M, \qquad T_k^e = \sigma_{jk}^M n_j;\\
&f_k^{e\,\text{env}} = \sigma_{jk,j}^{M\,\text{env}},\\
&T_k^{e\text{env}} = \sigma_{jk}^{M\,\text{env}} n_j^{\text{env}}.
\end{aligned} \tag{12}$$

Then the variational principle equation (6) is reduced to

$$\begin{aligned}
\delta\Pi' &= \delta\Pi_1' + \delta\Pi_2' - \delta W'^{\text{int}} = 0,\\
\delta\Pi_1' &= \int_{\overline{V}} \delta g\, dV - \delta W',\\
\delta\Pi_2' &= \int_{\overline{V}^{\text{env}}} \delta g^{\text{env}} dV - \delta W'^{\text{env}},\\
\delta W' &= \int_V \left(f_k + f_k^e - \rho\ddot{u}_k\right)\delta u_k dV - \int_V \rho_e \delta\varphi\, dV\\
&\quad + \int_{a_\sigma} \left(T_k^* - T_k^e\right)\delta u_k da - \int_{a_D} \sigma^* \delta\varphi\, da,\\
\delta W'^{\text{env}} &= \int_{V^{\text{env}}} \left(f_k^{\text{env}} + f_k^{e\,\text{env}} - \rho^{\text{env}}\ddot{u}_k^{\text{env}}\right)\delta u_k^{\text{env}} dV\\
&\quad - \int_{V^{\text{env}}} \rho_e^{\text{env}} \delta\varphi^{\text{env}} dV\\
&\quad + \int_{a_\sigma^{\text{env}}} \left(T_k^{\text{env}\,*} - T_k^{e\,\text{env}}\right)\delta u_k^{\text{env}} da\\
&\quad - \int_{a_D^{\text{env}}} \sigma^{\text{env}\,*} \delta\varphi^{\text{env}} da,\\
\delta W'^{\text{int}} &= \int_{a^{\text{int}}} \left(T_k^{\text{int}\,*} + T_k^{e\,\text{env}} - T_k^e\right)\delta u_k da\\
&\quad - \int_{a^{\text{int}}} \sigma^{\text{int}\,*} \delta\varphi\, da.
\end{aligned} \tag{13}$$

In (13) the variations of δu and $\delta\varphi$ are completely independent, that is, it is not needed to consider the migratory variation $\delta_u\varphi$. Equations (8)–(10) are reduced to

$$\begin{aligned}
&\sigma_{ij,i} + \left(f_j + f_j^e\right) = \rho\ddot{u}_j, \quad D_{i,i} = \rho_e, \quad \text{in } V,\\
&\sigma_{ij}n_i = T_j^* - T_j^{e*}, \quad \text{on } a_\sigma, \qquad D_in_i = -\sigma^*, \quad \text{on } a_D;\\
&\sigma_{ij,j}^{\text{env}} + \left(f_i^{\text{env}} + f_i^{e\,\text{env}}\right) = \rho\ddot{u}_i^{\text{env}}, \quad D_{i,i}^{\text{env}} = \rho_e^{\text{env}}, \quad \text{in } V^{\text{env}}\\
&\sigma_{ij}^{\text{env}}n_j^{\text{env}} = \left(T_i^{*\text{env}} - T_i^{*e\,\text{env}}\right), \quad \text{on } a_\sigma^{\text{env}},\\
&D_i^{\text{env}}n_i^{\text{env}} = -\sigma^{*\text{env}}, \quad \text{on } a_D^{\text{env}},\\
&\left(\sigma_{kl} - \sigma_{kl}^{\text{env}}\right)n_l = T_k^{*\,\text{int}} - T_i^e + T_i^{e\,\text{env}},\\
&D_kn_k - D_k^{\text{env}}n_k = -\sigma^{*\,\text{int}}, \quad \text{on } a^{\text{int}}.
\end{aligned} \tag{14}$$

3. The Static Electric Force Acting on a Dielectric

3.1. The Static Electric Force Acting on the Dielectric in a Capacitor. In any electromagnetic textbook we can find how to determine the static electric force acting on the dielectric in a capacitor consisted of two parallel electrode plates and filled dielectric with permittivity ϵ. Assume the length and width of the plates are infinite, the distance h between two plate electrodes is very small. The electric potential on the upper plate electrode is smaller than that on the lower plate.

Let the coordinate origin be located at its center, the plane x_1–x_3 is parallel, and axis x_2 is perpendicular to the electrode plates. There is no external force on the plate and inside the dielectric and the deformation is small. The electric field inside the dielectric of the capacitor is homogeneous and $\mathbf{E} = E_2\mathbf{n}$, $\mathbf{n} = \mathbf{i}_2$ due to that the plates are infinite, where $E_2 = \varphi/h$, φ is the difference of electric potentials between two electrodes. The static electric force acting on electrode plates can be obtained by the energy method.

In this simple case the static electric force can be directly derived from the general equation of the physical variational principle. According to (6) in this case we have

$$\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{\text{int}} = 0,$$
$$\delta\Pi_1 = \delta\int_V g\,dV = h\left[\sigma_{22}\delta u_{2,2} - D_2\delta E_2 - \left(\frac{1}{2}\right)E_2D_2\delta u_{2,2}\right],$$
$$\delta\Pi_2 = 0, \qquad \delta W^{\text{int}} = 0. \tag{15}$$

Give a virtual displacement under the constant electric potential on the electrode plate. It is noted that though φ is constant on the plate, but after virtual displacement φ is changed inside the dielectric. For a fixed x the change of the electric field due to changed φ is

$$\delta_\varphi E_2 = \frac{\varphi}{(h+\delta h)} - \frac{\varphi}{h} = -\frac{\varphi\delta h}{h^2}, \qquad -D_2\delta E_2 = D_2E_2\frac{\delta h}{h}. \tag{16}$$

Due to the electric potential φ on the electrode plate is constant, we have $\delta_u E_2 = E_{2,p}\delta u_p = 0$, so $\delta E_2 = \delta_\varphi E_2$. Therefore, finally we get

$$\delta\Pi = h\left[\sigma_{22}\delta u_{2,2} - D_2\delta E_2 - \left(\frac{1}{2}\right)E_2D_2\delta u_{2,2}\right]$$
$$= hD_2\left[\sigma_{22}\left(\frac{\delta h}{h}\right) + \left(\frac{1}{2}\right)\epsilon E_2^2\left(\frac{\delta h}{h}\right)\right] = 0 \tag{17}$$
$$\Longrightarrow T_2 = \sigma_{22} = -\frac{\epsilon E_2^2}{2}.$$

We can also use the Maxwell stress to derive the force acting on the dielectric directly. According to (10), we have $(S_{ij} - S_{ij}^{\text{env}})n_i = 0$ or $T_i = (\sigma_{ij} - \sigma_{ij}^{\text{env}})n_i = -\sigma_{ij}^M n_j$, where $\sigma_{ij}^{M\,\text{env}} = 0$ in the electrode is used. In the present case we have $T_2 = -\sigma_{22}^M = -D_2^2/2\epsilon$. If the force on the electrode plate is zero, then the static electric force acting on the dielectric is $-D_2^2/2\epsilon$, which is identical with that in (17).

In the electric textbooks the derivation is as follows: given the upper plate a virtual displacement δh along x_2, the electric charge on the electrode plate increases δq due to fixed φ, so the electric source supplies the energy $\varphi\delta q$. The energy in the dielectric increases $\delta(\varphi q/2) = \varphi\delta q/2$ and the remained energy $\varphi\delta q/2$ is used to overcome the work produced by the static electric force on the dielectric, that is, for virtual displacement δu we have

$$F\cdot\delta u = \sigma_{22}\delta h = \varphi\delta q - \left(\frac{1}{2}\right)\varphi\delta q = \left(\frac{1}{2}\right)\varphi\delta q$$
$$= \left(\frac{1}{2}\right)\varphi\delta(\epsilon E_2) = \left(\frac{\epsilon}{2}\right)\varphi\delta\left(\frac{\varphi}{h}\right) \tag{18}$$
$$= -\left(\frac{\epsilon}{2}\right)\left(\frac{\varphi}{h}\right)^2\delta h \Longrightarrow \sigma_{22} = -\left(\frac{\epsilon}{2}\right)E_2^2,$$

which is identical with that in (17). At least this result partly proves the general theory is correct.

3.2. The Static Electric Force in General Case. Though the static electric force acting on dielectric of a capacitor was derived in textbooks from the energy method as shown in above section, but the energy method did not be used to derive the Maxwell stress in the general case.

In the general case the Maxwell stress is introduced by the migratory variations of electric field. So the static electric force from (6) and (13) can be written as

$$\delta W^e = -\delta_u\Pi,$$
$$\delta_u\Pi = \int_V g_{,E}\cdot\delta_u E\,dV + \int_V g^e\delta u_{k,k}dV + \int_V \rho_e\delta_u\varphi\,dV$$
$$+ \int_{a_D}\sigma^*\delta_u\varphi\,da + \int_{V^{\text{env}}} g_{,E}^{\text{env}}\cdot\delta_u E^{\text{env}}dV$$
$$+ \int_{V^{\text{env}}} g^{e\,\text{env}}\delta u_{i,i}^{\text{env}}\,dV + \int_{V^{\text{env}}} \rho_e^{\text{env}}\delta_u\varphi^{\text{env}}\,dV$$
$$+ \int_{a_D^{\text{env}}}\sigma^{*\text{env}}\delta_u\varphi^{\text{env}}\,da + \int_{a^{\text{int}}}\sigma^{*\,\text{int}}\delta_u\varphi\,da,$$
$$\delta W^e = \int_V f_k^e\delta u_k\,dV + \int_{a_\sigma} T_k^e\delta u_k\,da + \int_{V^{\text{env}}} f_k^{e\,\text{env}}\delta u_k^{\text{env}}\,dV$$
$$+ \int_{a_\sigma^{\text{env}}} T_k^{e\,\text{env}}\delta u_k^{\text{env}}\,da + \int_{a^{\text{int}}}\left(T_k^e - T_k^{e\,\text{env}}\right)\delta u_k\,da. \tag{19}$$

It is easy to prove that f_k^e, T_k^e, $f_k^{e\,\text{env}}$, $T_k^{e\,\text{env}}$ derived from (19) are identical with that in (12), see Appendix B. If the environment is neglected, then we get

$$\int_V f_k^e\delta u_k\,dV + \int_{a_\sigma} T_k^e\delta u_k\,da$$
$$= -\left(\int_V g_{,E}\cdot\delta_u E\,dV + \int_V g^e\delta u_{k,k}\,dV\right. \tag{20}$$
$$\left. + \int_V \rho_e\delta_u\varphi\,dV + \int_{a_D}\sigma^*\delta_u\varphi\,da\right).$$

Using $\mathbf{D} = D_n\mathbf{n} + D_t\mathbf{t}$ and $\mathbf{E} = E_n\mathbf{n} + E_t\mathbf{t}$, and the continuous condition (10) and (11) of the $\mathbf{D}$, $\mathbf{E}$ on the interface, the Maxwell stress can also be rewritten as

$$
\begin{aligned}
&\mathbf{n}\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{env}}) = \widetilde{\mathbf{T}}^{*\,\text{int}},\\
&\widetilde{\mathbf{T}}^{*\,\text{int}} = \mathbf{T}^{*\,\text{int}} + \mathbf{n}\cdot\left(\boldsymbol{\sigma}^{M\,\text{env}} - \boldsymbol{\sigma}^{M}\right),\\
&\mathbf{n}\cdot(\sigma^{M\,\text{env}} - \sigma^{M}),\\
&= \left[\mathbf{n}\cdot(\mathbf{D}^{\text{env}}\otimes\mathbf{E}^{\text{env}}) - \left(\frac{1}{2}\right)(\mathbf{D}^{\text{env}}\cdot\mathbf{E}^{\text{env}})\mathbf{n}\right]\\
&\quad - \left[\mathbf{n}\cdot(\mathbf{D}\otimes\mathbf{E}) - \left(\frac{1}{2}\right)(\mathbf{D}\cdot\mathbf{E})\mathbf{n}\right]\\
&= \left(\frac{1}{2}\right)\left[D_n(E_n^{\text{env}} - E_n) - (D_t^{\text{env}} - D_t)E_t\right]\mathbf{n},
\end{aligned}
\tag{21}
$$

where $\mathbf{n}$ is the unit normal, subscripts n and t mean the normal and tangential direction respectively; there is no sum on n and t. Equation (21) points out that in the case of small strain the boundary surface traction corresponding to the Maxwell stress is along the normal direction.

4. Measurement of Material Constants in Isotropic Electrostrictive Materials

For isotropic materials we have

$$
\begin{aligned}
C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),\\
l_{ijkl} &= a_2\delta_{ij}\delta_{kl} + \left(\frac{a_1}{2}\right)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),\\
\epsilon_{ij} &= \epsilon\delta_{ij}, \qquad e_{kij} = 0.
\end{aligned}
\tag{22}
$$

So (3) and (4) are reduced to

$$
\begin{aligned}
g = {}&\left(\frac{1}{2}\right)\lambda\varepsilon_{ii}\varepsilon_{kk} + G\varepsilon_{ij}\varepsilon_{ij} - \left(\frac{1}{2}\right)\epsilon E_k E_k\\
&- \left(\frac{1}{2}\right)\left(a_2 E_i E_i \varepsilon_{kk} + a_1 E_i E_j \varepsilon_{ij}\right),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\sigma_{kl} &= \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - \left(\frac{1}{2}\right)(a_2 E_i E_i \delta_{kl} + a_1 E_k E_l),\\
D_k &= \epsilon E_k + (a_2 E_k \varepsilon_{mm} + a_1 E_l \varepsilon_{kl}) \approx \epsilon E_k,\\
S_{kl} &= \sigma_{kl} + \sigma_{kl}^{M} = \lambda\varepsilon_{ii}\delta_{kl} + 2G\varepsilon_{kl} - \left(\frac{1}{2}\right)(a_2 + \epsilon)E_m E_m \delta_{kl}\\
&\quad + \left(\frac{1}{2}\right)(2\epsilon - a_1)E_k E_l.
\end{aligned}
\tag{24}
$$

The first formula in (24) is just the usual form of the constitutive equation, where a_1 and a_2 are known as electrostrictive coefficients. From (8)–(11), it is known that solving $\mathbf{S}$ is easier than solving $\boldsymbol{\sigma}$, so in experiments the measured variables usually are $(\mathbf{S}, \boldsymbol{\varepsilon}, \mathbf{E})$. So that in usual experiments the measured material coefficients are $2\epsilon - a_1$ and $a_2 + \epsilon$. Therefore, when we do experiments we should clearly provide what variables are used. In isotropic materials or materials without the electromagnetic body couple variables $\mathbf{S}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}^M$ are all symmetric.

As an example, we discuss the electroelastic field of an isotropic rectangular dielectric with material constants $\boldsymbol{\epsilon}$, a_1, a_2. The length, width and height of the dielectric are $\mathbf{l}$, b, and h, respectively. Let the coordinate origin be located at its center, the axes are parallel to its edges, the axis x_2 is perpendicular to its middle plane. Assume the electric field in dielectric is homogeneous $\mathbf{E} = E_2\mathbf{n}$, $\mathbf{n} = \mathbf{i}_2$ and the dielectric is free from the external force $(\mathbf{f}, \mathbf{T}^{*\,\text{int}} = \mathbf{0})$.

4.1. The Surrounding of the Dielectric Is Air. In the air there is no mechanical stress. On the interface we have

$$
\begin{aligned}
&E_t^{\text{air}} = E_t, \quad D_n^{\text{air}} = D_n \quad \text{on interface; or}\\
&E_1^{\text{air}} = E_1 = 0, \qquad E_3^{\text{air}} = E_3 = 0,\\
&D_2^{\text{air}} = D_2\left(\epsilon_0 E_2^{\text{air}} = \epsilon E_2\right), \qquad x_2 = \pm\frac{h}{2},\\
&E_1^{\text{air}} = E_1 = 0, \qquad E_3^{\text{air}} = E_3 = 0,\\
&E_2^{\text{air}} = E_2; \qquad x_1 = \pm\frac{l}{2}, \quad \text{or} \quad x_3 = \pm\frac{b}{2},
\end{aligned}
\tag{25}
$$

where n and t mean the normal and tangential directions, respectively. In this case from (8)–(10) we get

$$
\begin{aligned}
&S_{ij,j} = 0, \quad \text{in } V,\\
&\left(S_{ij} - S_{ij}^{\text{air}}\right)n_i = 0, \quad \text{on } a^{\text{int}} \Longrightarrow \sigma_{ij} = \left(\sigma_{ij}^{M\,\text{air}} - \sigma_{ij}^{M}\right)^{\text{int}},\\
&\sigma_{22}^{M} = \left(\frac{1}{2}\right)D_2E_2, \quad \sigma_{11}^{M} = \sigma_{33}^{M} = -\left(\frac{1}{2}\right)D_2E_2,\\
&\sigma_{ij}^{M} = 0, \quad \text{when } i \neq j; \quad \text{on } a^{\text{int}},\\
&\sigma_{22}^{M\,\text{air}} = \left(\frac{1}{2}\right)D_2^{\text{air}}E_2^{\text{air}}, \quad \sigma_{11}^{M\,\text{air}} = \sigma_{33}^{M\,\text{air}} = -\left(\frac{1}{2}\right)D_2^{\text{air}}E_2^{\text{air}};\\
&\sigma_{ij}^{M\,\text{air}} = 0, \quad \text{when } i \neq j, \quad \text{on } a^{\text{int}} \Longrightarrow\\
&\sigma_{11} = \sigma_{33} = -\left(\frac{1}{2}\right)\left[\left(\frac{\epsilon^2}{\epsilon_0}\right) - \epsilon\right]E_2^2,\\
&\sigma_{22} = \left(\frac{1}{2}\right)\left[\left(\frac{\epsilon^2}{\epsilon_0}\right) - \epsilon\right]E_2^2.
\end{aligned}
\tag{26}
$$

Using the constitutive equation (24) we get

$$
\begin{aligned}
-\left(\frac{1}{2}\right)\left[\left(\frac{\epsilon^2}{\epsilon_0}\right) - \boldsymbol{\epsilon}\right]E_2^2 &= (3\lambda\theta + 2G\varepsilon_{11}) - \left(\frac{1}{2}\right)a_2E_2^2,\\
\left(\frac{1}{2}\right)\left[\left(\frac{\epsilon^2}{\epsilon_0}\right) - \boldsymbol{\epsilon}\right]E_2^2 &= (3\lambda\theta + 2G\varepsilon_{22}) - \left(\frac{1}{2}\right)(a_2 + a_1)E_2^2,\\
-\left(\frac{1}{2}\right)\left[\left(\frac{\epsilon^2}{\epsilon_0}\right) - \boldsymbol{\epsilon}\right]E_2^2 &= (3\lambda\theta + 2G\varepsilon_{33}) - \left(\frac{1}{2}\right)a_2E_2^2.
\end{aligned}
\tag{27}
$$

4.2. The Dielectric and Air Are All between Two Parallel Infinite Rigid Electrodes. In electrodes $\mathbf{E} = \mathbf{0}$, so there is no Maxwell stress. In this case from (8)–(10) we get

$$S_{ij,j} = 0, \quad \text{in } V,$$
$$\left(S_{ij} - S_{ij}^{\text{env}}\right) n_i = 0, \quad \text{on } a^{\text{int}} \Longrightarrow \sigma_{ij} = \left[\sigma_{ij}^{M\text{ env}} - \sigma_{ij}^{M}\right]^{\text{int}},$$
$$\sigma_{22}^{M} = \left(\frac{1}{2}\right)\epsilon E_2^2, \quad \sigma_{11}^{M} = \sigma_{33}^{M} = -\left(\frac{1}{2}\right)\epsilon E_2^2,$$
$$\sigma_{ij}^{M} = 0, \quad \text{when } i \neq j, \text{ on } a^{\text{int}},$$
$$\sigma_{22}^{M\text{ plate}} = 0, \quad \sigma_{11}^{M\text{ air}} = \sigma_{33}^{M\text{ air}} = -\left(\frac{1}{2}\right)\epsilon_0 E_2^2,$$
$$\sigma_{ij}^{M\text{ env}} = 0, \quad \text{when } i \neq j, \quad \text{on } a^{\text{int}} \Longrightarrow$$
$$\sigma_{11} = \sigma_{33} = \left(\frac{1}{2}\right)(\epsilon - \epsilon_0)E_2^2,$$
$$\sigma_{22} = -\left(\frac{1}{2}\right)\epsilon E_2^2. \tag{28}$$

Using the constitutive equation (24) we get

$$\left(\frac{1}{2}\right)(\epsilon - \epsilon_0)E_2^2 = (3\lambda\theta + 2G\varepsilon_{11}) - \left(\frac{1}{2}\right)a_2 E_2^2,$$
$$-\left(\frac{1}{2}\right)\epsilon E_2^2 = (3\lambda\theta + 2G\varepsilon_{22}) - \left(\frac{1}{2}\right)(a_2 + a_1)E_2^2, \tag{29}$$
$$\left(\frac{1}{2}\right)(\epsilon - \epsilon_0)E_2^2 = (3\lambda\theta + 2G\varepsilon_{33}) - \left(\frac{1}{2}\right)a_2 E_2^2.$$

In (27) and (29) $3\theta = \varepsilon_{ii}$. In the discussed cases the third equation in (27) and (29) can be omitted. In experiments we can measure strains from the given electric field. From (27) or (29) or other improved methods we can get electrostrictive coefficients. It is clear that the environment has obviously effect.

The above example is very simple and ideal. The real experimental set is more complex, but from above discussions, we can see that in order to get correct electrostrictive constants in experiments we need consider the entire system including the dielectric medium, its environment and their common boundary.

5. Conclusions

In this paper, we discuss the nonlinear electroelastic analysis in the electrostrictive materials by the physical variational principle. Given a general expression of the electric force. Using the governing equation proposed in this paper we give a possible method to correctly measure the electrostrictive coefficients. It is shown that in order to get correct material constants in experiments we need take the correct governing equations and consider the entire system including the dielectric medium, its environment and their common boundary.

Appendices

A. The Derivation of $\delta\mathbf{E}$ in (7)

In our previous papers $\delta\mathbf{E}$ in (7) was directly given, here we shall give a simple derivation.

Assume the variational functional is

$$I = \int_V L\left(x_i, u_\alpha, u_{\alpha,j}\right) dV, \tag{A.1}$$

where $\mathbf{x}$, $\mathbf{u}$, $\mathbf{u}_{,j}$ are the independent variable, dependent variable, and the derivative of the dependent variable, respectively. Assume an infinitesimal transformation

$$x_i \longrightarrow x_i' = x_i + \delta u_i\left(x_j, \varphi\right),$$
$$\varphi(x_i) \longrightarrow \varphi'\left(x_i'\right) = \varphi(x_i) + \delta\varphi(x_i, \varphi),$$
$$\delta\varphi = \varphi'\left(x_i'\right) - \varphi(x_i) \tag{A.2}$$
$$= \left[\varphi(x_i + \delta u_i) + \delta_\varphi \varphi(x_i)\right] - \varphi(x_i)$$
$$= \delta_\varphi \varphi + \delta_u \varphi = \delta_\varphi \varphi + \varphi_{,i}\delta u_i.$$

Noting

$$\frac{\partial x_i'}{\partial x_j} \approx \delta_{ij} + \frac{\partial \delta u_i}{\partial x_j}, \quad j = \left|\frac{\partial x_i'}{\partial x_j}\right| \approx 1 + \frac{\partial \delta u_i}{\partial x_i}, \quad \frac{\partial x_i}{\partial x_j'} \approx \delta_{ij} - \frac{\partial \delta u_i}{\partial x_j}, \tag{A.3}$$

from (A.2) we get

$$\delta\varphi_{,j} = \frac{\partial \varphi'\left(x_i'\right)}{\partial x_j'} - \frac{\partial \varphi(x_i)}{\partial x_j}$$
$$= \frac{\partial\left[\varphi(x_i) + \delta\varphi(x_i, \varphi)\right]}{\partial x_k}\frac{\partial x_k}{\partial x_j'} - \frac{\partial \varphi(x_i)}{\partial x_j}$$
$$= \frac{\partial \delta\varphi(x_i, \varphi)}{\partial x_j} - \frac{\partial \varphi(x_i)}{\partial x_k}\frac{\partial \delta u_k}{\partial x_j}$$
$$= \frac{\partial\left(\delta_\varphi \varphi + \varphi_{,i}\delta u_i\right)}{\partial x_j} - \frac{\partial \varphi(x_i)}{\partial x_k}\frac{\partial \delta u_k}{\partial x_j} \tag{A.4}$$
$$= \frac{\partial\left(\delta_\varphi \varphi\right)}{\partial x_j} + \frac{\partial(\varphi_{,i}\delta u_i)}{\partial x_j} - \frac{\partial \varphi(x_i)}{\partial x_k}\frac{\partial \delta u_k}{\partial x_j}$$
$$= \frac{\partial\left(\delta_\varphi \varphi\right)}{\partial x_j} + \frac{\partial(\varphi_{,i})}{\partial x_j}\delta u_i.$$

Equation (A.4) is identical with $\delta\mathbf{E}$ in (7) in text.

B. The Derivation of (19)

From (19) we have

$$\begin{aligned}
\delta_u \Pi = {} & \int_V g_E \cdot \delta_u E \, dV + \int_V g^e \delta u_{k,k} \, dV + \int_V \rho_e \delta_u \varphi \, dV \\
& + \int_{a_D} \sigma^* \delta_u \varphi \, da + \int_{V^{\text{env}}} g_E^{\text{env}} \cdot \delta_u E^{\text{env}} \, dV \\
& + \int_{V^{\text{env}}} g^{e\,\text{env}} \delta u_{i,i}^{\text{env}} \, dV + \int_{V^{\text{env}}} \rho_e^{\text{env}} \delta_u \varphi^{\text{env}} \, dV \\
& + \int_{a_D^{\text{env}}} \sigma^{*\text{env}} \delta_u \varphi^{\text{env}} \, da + \int_{a^{\text{int}}} \sigma^{*\,\text{int}} \delta_u \varphi \, da \\
= {} & -\int_V D_i E_{p,i} \delta u_p \, dV - \int_V \left(\frac{1}{2}\right) D_k E_k \delta u_{j,j} \, dV \\
& + \int_V \rho_e \delta_u \varphi \, dV + \int_{a_D} \sigma^* \delta_u \varphi \, da - \int_{V^{\text{env}}} D_i^{\text{env}} \delta_u E_i^{\text{env}} \, dV \\
& - \int_{V^{\text{env}}} \left(\frac{1}{2}\right) D_k^{\text{env}} E_k^{\text{env}} \delta u_{j,j}^{\text{env}} \, dV + \int_{V^{\text{env}}} \rho_e^{\text{env}} \delta_u \varphi^{\text{env}} \, dV \\
& + \int_{a_D^{\text{env}}} \sigma^{*\text{env}} \delta_u \varphi^{\text{env}} \, da + \int_{a^{\text{int}}} \sigma^{*\,\text{int}} \delta_u \varphi \, da \\
= {} & -\int_V \left(D_i E_p\right)_{,i} \delta u_p \, dV + \int_V \left(\rho_e - D_{i,i}\right) \delta_u \varphi \, dV \\
& - \left(\frac{1}{2}\right) \int_a D_k E_k n_j \delta u_j \, da + \left(\frac{1}{2}\right) \int_V \left(D_k E_k\right)_{,j} \delta u_j \, dV \\
& + \int_{a_D} \sigma^* \delta_u \varphi \, da - \int_{V^{\text{env}}} \left(D_i^{\text{env}} E_p^{\text{env}}\right)_{,i} \delta u_p^{\text{env}} \, dV \\
& + \int_{V^{\text{env}}} \left(\rho_e^{\text{env}} - D_{i,i}^{\text{env}}\right) \delta_u \varphi^{\text{env}} \, dV \\
& - \left(\frac{1}{2}\right) \int_{V^{\text{env}}} D_k^{\text{env}} E_k^{\text{env}} n_j^{\text{env}} \delta u_j^{\text{env}} \, dV \\
& + \left(\frac{1}{2}\right) \int_{V^{\text{env}}} \left(D_k^{\text{env}} E_k^{\text{env}}\right)_{,j} \delta u_j^{\text{env}} \, dV \\
& + \int_{a_D^{\text{env}}} \sigma^{*\text{env}} \delta_u \varphi^{\text{env}} \, da + \int_{a^{\text{int}}} \sigma^{*\,\text{int}} \delta_u \varphi \, da.
\end{aligned} \tag{B.1}$$

Using (8)–(11), that is,

$$\begin{aligned}
D_i n_i + \sigma^* &= 0, \quad \text{on } a_D, \\
D_{i,i} - \rho_e &= 0, \quad \text{in } V, \\
\left(D_i - D_i^{\text{env}}\right) n_i &= -\sigma^{*\,\text{int}}, \quad \text{on } a^{\text{int}}
\end{aligned} \tag{B.2}$$

and adding terms $\int_a D_i n_i (E_p \delta u_p + \delta_u \varphi) da = 0$ and $\int_{a^{\text{env}}} D_i^{\text{env}} n_i^{\text{env}} (E_p^{\text{env}} \delta u_p^{\text{env}} + \delta_u \varphi^{\text{env}}) da = 0$ to (B.1), then (B.1) is reduced to

$$\begin{aligned}
\delta_u \Pi = {} & -\int_V \left(D_i E_p\right)_{,i} \delta u_p \, dV - \int_V \left(\frac{1}{2}\right) D_k E_k n_j \delta u_j \, dV \\
& + \int_V \left(\frac{1}{2}\right) \left(D_k E_k\right)_{,j} \delta u_j \, dV + \int_a D_i E_p n_i \delta u_p \, da \\
& - \int_{V^{\text{env}}} \left(D_i^{\text{env}} E_p^{\text{env}}\right)_{,i} \delta u_p^{\text{env}} \, dV \\
& - \int_{V^{\text{env}}} \left(\frac{1}{2}\right) D_k^{\text{env}} E_k^{\text{env}} n_j^{\text{env}} \delta u_j^{\text{env}} \, dV \\
& + \int_{V^{\text{env}}} \left(\frac{1}{2}\right) \left(D_k^{\text{env}} E_k^{\text{env}}\right)_{,j} \delta u_j^{\text{env}} \, dV \\
& + \int_{a^{\text{env}}} D_i^{\text{env}} E_p^{\text{env}} n_i^{\text{env}} \delta u_p^{\text{env}} \, da \\
= {} & \int_{a_\sigma} \sigma_{ij}^M n_i \delta u_j \, da - \int_V \sigma_{ij,i}^M \delta u_j \, dV \\
& + \int_{a_\sigma^{\text{env}}} \sigma_{ij}^{M\,\text{env}} n_i^{\text{env}} \delta u_j^{\text{env}} \, da - \int_V \sigma_{ij,i}^{M\,\text{env}} \delta u_j^{\text{env}} \, dV \\
& + \int_{a^{\text{int}}} \sigma_{ij}^M n_i \delta u_j \, da + \int_{a^{\text{int}}} \sigma_{ij}^{M\,\text{env}} n_i^{\text{env}} \delta u_j^{\text{env}} \, da.
\end{aligned} \tag{B.3}$$

From (19), we know that he work done by the equivalent static electric force is

$$\begin{aligned}
\delta W^e = {} & \int_V f_k^e \delta u_k \, dV + \int_{a_\sigma} T_k^e \delta u_k \, da + \int_{V^{\text{env}}} f_k^{e\,\text{env}} \delta u_k^{\text{env}} \, dV \\
& + \int_{a_\sigma^{\text{env}}} T_k^{e\,\text{env}} \delta u_k^{\text{env}} \, da + \int_{a^{\text{int}}} \left(T_k^e - T_k^{e\,\text{env}}\right) \delta u_k \, da.
\end{aligned} \tag{B.4}$$

From (B.3) and (B.4), we immediately get (19).

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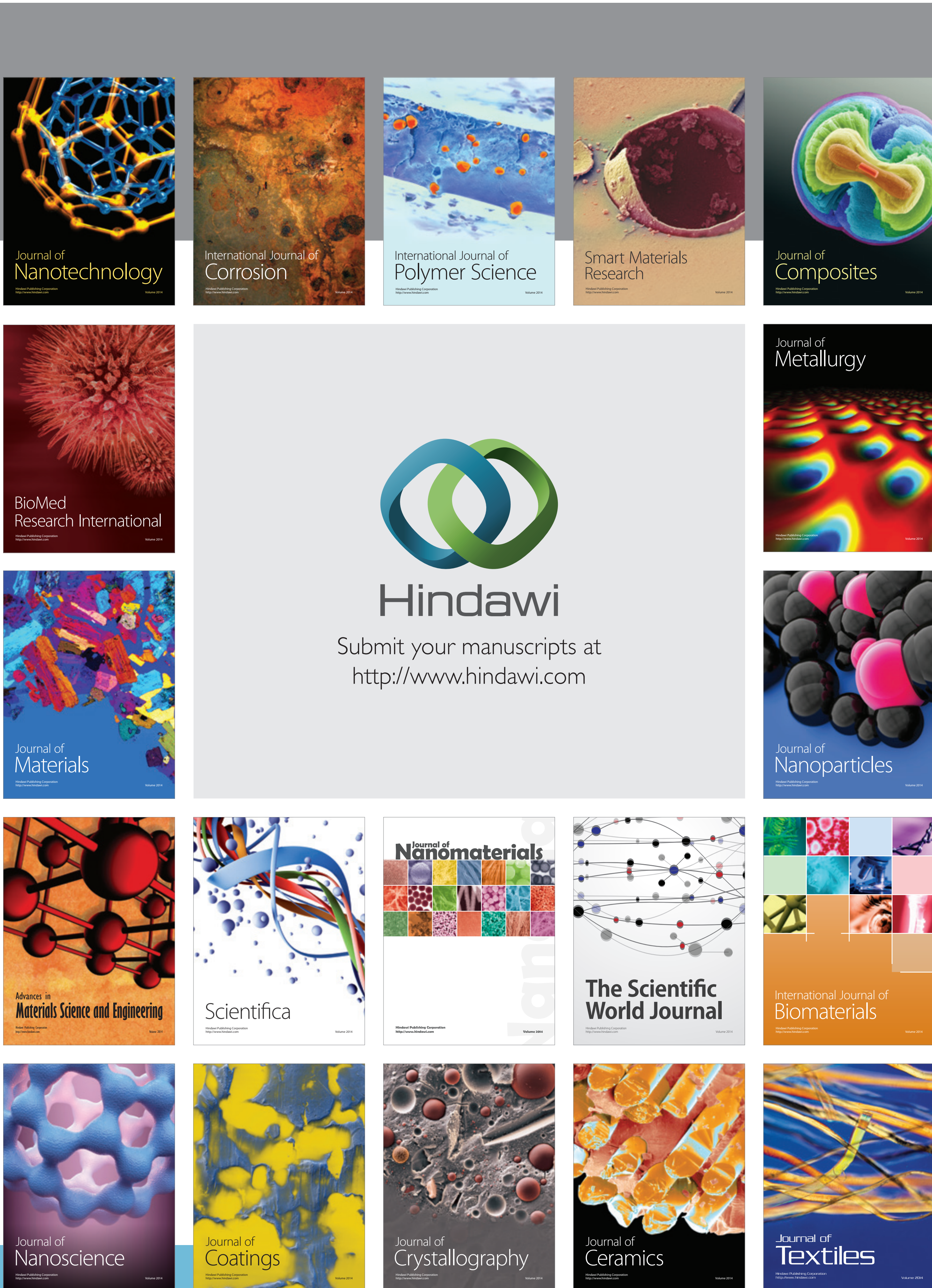
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